Name:…………………………………………………………………………………….

Student Number: ………………………………………………………………………..

Test 4 on WPPH16001.2018-2019
"Electricity and Magnetism"

Content: 10 pages (including this cover page)

Friday May 3 2019; A. Jacobshal 01, 9:00-11:00

- Write your full name and student number in the place above
- Write your answers in the designated areas
- Read the questions carefully
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Do not use a red pen (it's used for grading) or a pencil
- Books, notes, phones, and tablets are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

The weighting of the questions:

	Maximum points	Points scored
Question 1	10	
Question 2	9	
Question 3	11	
Question 4	10	
Total	**40**	

Grade = 1 + 9 x (score/max score).

Grade: _________

Question 1. (10 points)

An alternating current $I(t) = I_o cos\omega t$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

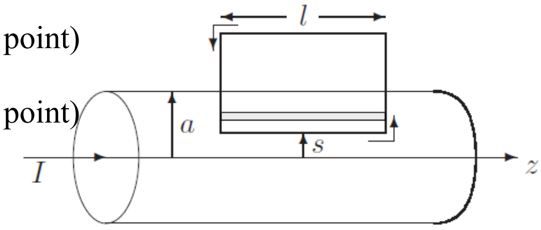
1. In what direction does the induced electric field point? (2 points)
2. Assuming that the field goes to zero as the distance from the axis goes to infinity, find the electric field as a function of coordinates and time. (6 points)
3. Find the displacement current density $\boldsymbol{J}_d$ (2 points)

<u>Answers</u>

Model answers (Griffiths, Problem 7.15; Problem 7.36) (10 points)

1. (2 points)
The magnetic field (in the quasistatic approximation) is "circumferential". This is analogous to the current in a solenoid, and hence the electric field is longitudinal (1 point).
The direction is along z: $\mathbf{E} \parallel \hat{\mathbf{l}}$. (1 point)

2. (6 points)
Use the "amperian loop" shown (1 point)



Outside, $\mathbf{B} = \mathbf{0}$, so here $\mathbf{E} = \mathbf{0}$ (like $\mathbf{B}$ outside a solenoid) (1 point)

$$\oint \mathbf{E} \cdot d\mathbf{l} = El = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{a} = -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'} l\, ds'$$

$$E = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{a}{s}\right)$$

(2 points)

$$\frac{dI}{dt} = -I_0\omega\, sin(\omega t)$$

$$\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi}\ln\left(\frac{a}{s}\right)\sin(\omega t)\ \hat{\mathbf{z}}$$

(2 points)

3. (2 points)

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

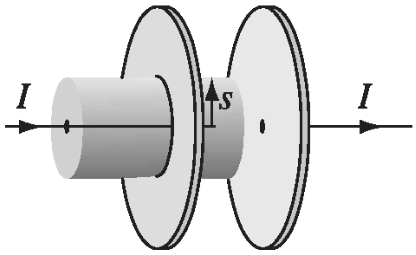
(1 point)

$$\mathbf{J}_d = \epsilon_0 \frac{\mu_0 I_0 \omega^2}{2\pi}\ln\left(\frac{a}{s}\right)\cos(\omega t)\,\hat{\mathbf{z}}$$

(1 point)

Question 2. (9 points)

The constant current I runs through thin wires that connect to the centers of the plates that form a parallel-plate capacitor (see Figure). The radius of the capacitor is a, and the separation of the plates is $w<<a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at time $t = 0$.



1. Find the electric field between the plates, as a function of time t. (3 points)

2. Using the cylindrical surface with radius s as depicted in Figure, which is open at the right end and extends to the left through the plate and terminates outside the capacitor, find the magnetic field between the plates at a distance s from the axis. (6 points)

Tip: Note that the displacement current through this surface is zero, and there are two contributions to I_{enc}: incoming current I and outcoming current due to redistribution of charges over the capacitor plate.

Answers

Model answers (Problem 7.35/7.34): **(9 points)**

1. (3 points)

$$\mathbf{E} = \frac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}}$$

(1 point)

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$$

(1 point)

$$\mathbf{E} = \frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}$$

(1 point)

2. (6 points)

A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius s. The charge density (at time t) is

$$\sigma(t) = \frac{[I - I(s)]t}{\pi s^2}$$

Since we are told this is independent of s, it must be that

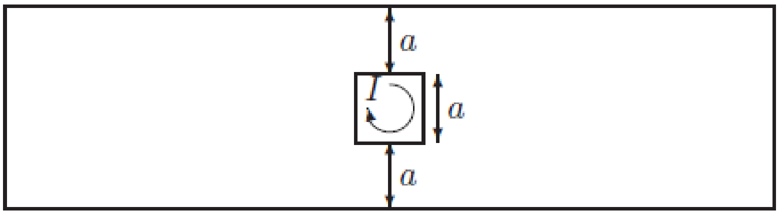
$$\frac{[I - I(s)]}{s^2} = \beta$$

where β is a constant. But $I(a) = 0$, so $\beta a^2 = I$, or $\beta = I/a^2$ and $I(s) = I(1 - s^2/a^2)$

$$B\,2\pi s = \mu_0 I_{enc} = \mu_0[I - I(s)] = \mu_0 I\frac{s^2}{a^2} \;\Rightarrow\; \mathbf{B} = \frac{\mu_0 I}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}$$

Question 3. (11 points)

A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the small square loop is gradually increasing: $dI/dt=k$ (k is a constant).



1. Find the mutual inductance of the loops. (6 points)
2. Find the emf induced in the big loop. (2 points)
3. Which way will the induced current flow? (3 points)

Answers

Model answers (Problem 7.23): (11 points)

1. (6 points)

It is hard to calculate M using a current in the little loop, so, exploiting the *equality of the mutual inductances*, we will find the flux through the little loop when a current I flows in the big loop

$\Phi = MI$ (2 points)

The field of one long wire is $B = \frac{\mu_0 I}{2\pi s}$

$$\Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s} a\, ds = \frac{\mu_0 I a}{2\pi} \ln 2$$

(2 points)

So that the total flux is

$$\Phi = 2\Phi_1 = \frac{\mu_0 I a \ln 2}{\pi} \quad \Rightarrow \quad M = \frac{\mu_0 a \ln 2}{\pi}$$

(2 points)

2. (2 points)

$$\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{d\text{I}}{dt} = -Mk$$

$$\mathcal{E} = \frac{\mu_0 k a \ln 2}{\pi}$$

3. (3 points)

Direction: Field lines point in, for the inside of the little loop, and out everywhere outside the little loop. The big loop encloses all of the former, and only part of the latter, so net flux is inward. Therefore, the net flux (through the big loop), due to I in the little loop, is into the page. (2 points)

This flux is increasing, so the induced current in the big loop is such that its field points out of the page: it flows counterclockwise. (1 point)

Question 4. (10 points)

1. Find the self-inductance $\mathcal{L}$ per unit length of a long solenoid, of radius R, carrying n turns per unit length. (3 points)
2. Find the energy stored in a section of length l of such a solenoid calculated via self-inductance. (3 points)
3. Now find the energy stored in a section of length l of such a solenoid using the expression for the energy via magnetic field. (3 points)
4. Do the results of question 2 and 3 make sense? (1 point)

Answers

Model answers (Problem 7.24 and 7.28): **(10 points)**

1. (3 points)

$B = \mu_0 n I \quad \Rightarrow$

Flux through a *single* turn $\Phi_1 = \mu_0 n I \pi R^2$

In a length *l* there are *nl* such turns, so that total flux

$\Phi = \mu_0 n^2 \pi R^2 I l$

The definition of self-inductance

$\Phi = LI$

so that self-inductance per unit length is

$\mathcal{L} = \mu_0 n^2 \pi R^2$

2. (3 points)

$$W = \frac{1}{2} L I^2; L = \mu_0 n^2 \pi R^2 l \quad \Rightarrow \quad W = \frac{1}{2} \mu_0 n^2 \pi R^2 l I^2$$

3. (3 points)

$$W = \frac{1}{2\mu_0} \int B^2 d\tau$$

$B = \mu_0 n I$ inside and zero outside; $\int d\tau = \pi R^2 l$

$$W = \frac{1}{2\mu_0} \mu_0^2 n^2 I^2 \pi R^2 l = \frac{1}{2} \mu_0 n^2 \pi R^2 l I^2$$

4. Yes, the energy is the same. Our equations work! (1 point)

Maxim Pchenitchnikov
April 30 2018

Steven Hoekstra